Version JL/2

EGT3 / EGT2
ENGINEERING TRIPOS PART IIB / ENGINEERING TRIPOS PART IIA

Wednesday 22 April 2015   2.00 to 3.30

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Consider the Poisson equation

$$\nabla^2 u = \delta \ , \tag{1}$$

where δ is the three-dimensional delta function which satisfies

$$\delta = 0, \quad \boldsymbol{x} \neq 0,$$
$$\int \delta dV = 1 \ ,$$

for any volume including the origin. Show that the trial solution

$$u = -\frac{1}{4\pi r}, \ \ r = |\boldsymbol{x}| \ , \tag{2}$$

satisfies $\nabla^2 u = 0$ for $\boldsymbol{x} \neq 0$ as well as the integral condition

$$\oint_{S_R} \nabla u \cdot d\boldsymbol{S} = 1 \ ,$$

where S_R is a spherical control surface of radius R, centered at the origin. Hence confirm that the trial solution (2) is indeed the solution of (1) in an infinite domain. [40%]

(b) Let $\boldsymbol{A}$ be the vector potential of the static magnetic field $\boldsymbol{B}$, where $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ and $\nabla \cdot \boldsymbol{A} = 0$. The distribution of $\boldsymbol{B}$ is governed by Ampère's law,

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \ ,$$

where μ_0 is the permeability of free space and $\boldsymbol{J}$ the current density. Confirm that

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J} \tag{3}$$

and use the result of (a), along with superposition, to derive the Green's function solution to (3) :

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} dV' \ .$$

[30%]

(c) Now show that this Green's function solution is equivalent to the Biot-Savart law

$$\boldsymbol{B} = -\frac{\mu_0}{4\pi} \int \frac{\boldsymbol{r} \times \boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{r}|^3} dV', \ \ \boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}' \ .$$

You may use the identity

$$\nabla \times \left[\frac{\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \right] = \nabla \left[\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \right] \times \boldsymbol{J}(\boldsymbol{x}') \ .$$

where ∇ operates on $\boldsymbol{x}$, not on $\boldsymbol{x}'$. [30%]

2 (a) Consider a beam of flexural rigidity EI and mass per unit length ρA. It sits on an elastic foundation which exerts a restoring force per unit length of $S\eta$, where S is a constant, and $\eta(x,t)$ the vertical displacement of the beam. The equation of motion is

$$EI\frac{\partial^4\eta}{\partial x^4} + S\eta + \rho A\frac{\partial^2\eta}{\partial t^2} = 0 ,$$

which supports flexural vibrations. By considering a solution of the form

$$\eta = \exp[i(kx - \omega t)] ,$$

show that the ratio of the group velocity c_g to phase speed c_p is

$$\frac{c_g}{c_p} = \frac{2EIk^4}{EIk^4 + S} .$$

[35%]

(b) Show that the group velocity of a wave packet is equal to the phase speed when the dominant wavelength satisfies

$$k_0 = (S/EI)^{1/4} .$$

A beam of infinite length is subject to a localised initial displacement of dominant wavenumber k. Describe the resulting wave pattern distinguishing between the speed of the wave crests and that of the overall wave packets. Consider both the case $k > k_0$ and $k < k_0$. [35%]

(c) A flat plate of flexural rigidity D sits on an elastic foundation of stiffness S and its transverse displacement $\eta(x,y,t)$ is governed by

$$D\nabla^4\eta + S\eta + \rho A\frac{\partial^2\eta}{\partial t^2} = 0 ,$$

where ∇^4 is the biharmonic operator

$$\nabla^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 .$$

Show that the ratio of the group velocity to phase speed is now

$$\frac{\boldsymbol{c}_g}{c_p} = \frac{2Dk^3\boldsymbol{k}}{Dk^4 + S}, \quad k = |\boldsymbol{k}| ,$$

where $\boldsymbol{k}$ is the wavevector. In what way is the direction of wave energy propagation fundamentally different to that of inertial waves in a rapidly-rotating fluid or internal gravity waves in a stratified fluid? [30%]

3 We wish to determine the stationary function $u(x)$ for

$$I(u) = \int_0^1 (u')^2 dx, \;\; u(0) = 0 \; ,$$

subject to the constraint

$$\int_0^1 u^2 dx = 1 \; ,$$

using the Lagrangian multiplier method. Note that $u(1)$ is not specified, but rather $u(1)$ is to be determined along with the stationary function.

(a) Write down the augmented integrand of the constrained functional. [10%]

(b) Deduce the Euler equation for the augmented functional. [10%]

(c) What is the boundary condition for $u(x)$ at $x = 1$? [20%]

(d) Deduce the stationary function $u(x)$ of this question. [60%]

4 Consider the differential equation

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) = -1, \tag{4}$$

with boundary conditions

$$u(1) = 0, \quad u(2) = -1.$$

Suppose the interval $[1,2]$ is partitioned in n uniform cells $x_0 = 1 < x_1 < \ldots < x_i < \ldots < x_n = 2$ such that $x_{i+1} - x_i = h$ is constant. The piecewise linear ‘hat-like’ function ϕ_i is defined as $\phi_i(x_j) = \delta_{ij}$, where δ_{ij} is Kronecker delta, and $i,\ j = 0, \cdots, n$.

(a) Deduce the weak form of Eqn. (4). [15%]

(b) Deduce the equivalent variational form of Eqn. (4). [10%]

(c) For $i,\ j = 0, \cdots, n$, calculate

$$K_{ij} = \int_1^2 x\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx\ .$$

[25%]

(d) Suppose $n = 2$. Calculate the approximate solution for $u(x)$ using the Galerkin method with the trial function

$$\bar{u} = c_0\phi_0 + c_1\phi_1 + c_2\phi_2\ ,$$

where c_0, c_1 and c_2 are constants to be determined. [30%]

(e) Compare the above approximate solution with the exact solution of Eqn. (4) and explain your result. [20%]

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Module 4M12: Partial differential equations and variational methods, 2015
Numeric value of Q4, (e) : $u_1 = 0$, $u_1 = -1/2$, $u_2 = -1$.