EGT3 / EGT2
ENGINEERING TRIPOS PART IIB / ENGINEERING TRIPOS PART IIA

Tuesday 3 May 2016   2.00 to 3.30

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Explain the difference between dispersive and non-dispersive waves and give a physical interpretation of d'Alembert's solution to the wave equation. [20%]

(b) Consider a slowly-modulated, one-dimensional wave train of the form

$$\eta(x,t) = A(x,t)\exp[i\theta(x,t)] \ ,$$

where A is the local amplitude and θ the phase function. The local wave-number and frequency of the wave, $k(x,t)$ and $\omega(x,t)$, are defined as

$$k = \partial\theta/\partial x, \quad \omega = -\partial\theta/\partial t \ .$$

If A, k and ω are slowly varying functions of x and t, satisfying the dispersion relation $\omega = \omega(k)$, show that an observer must move at the speed $c_g(k) = d\omega/dk$ to keep seeing waves of wavelength $\lambda = 2\pi/k$. [30%]

(c) The governing equation for two-dimensional gravity waves in a uniformly stratified fluid is

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)u_z + N^2\frac{\partial^2 u_z}{\partial x^2} = 0 \ ,$$

where N is a constant and u_z is the vertical velocity of the fluid. Derive an expression for the dispersion relationship for such waves and determine the range of realisable frequencies. [15%]

(d) Show that the group velocity for two-dimensional gravity waves is given by an expression of the form

$$\boldsymbol{c}_g = \pm\frac{Nk_z}{|\boldsymbol{k}|^3}\boldsymbol{f}(k_x,k_z,\hat{\boldsymbol{e}}_x,\hat{\boldsymbol{e}}_z) \ ,$$

where $\boldsymbol{k} = k_x\hat{\boldsymbol{e}}_x + k_z\hat{\boldsymbol{e}}_z$ is the wave-vector, $\hat{\boldsymbol{e}}_x$ and $\hat{\boldsymbol{e}}_z$ unit vectors, and $\boldsymbol{f}$ some unknown vector function. Find $\boldsymbol{f}$. [35%]

2 Heat flows radially outward into an infinite solid from a point source located at the origin. The governing equation for the spherically symmetric temperature field, $T(r,t)$, is

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{\dot{Q}}{\rho c_p}\delta \ ,$$

where $\dot{Q}$ is the rate of heat release from the origin, α the thermal diffusivity, ρ the density of the solid, c_p the specific heat capacity, and δ the three-dimensional delta function which satisfies $\int \delta \ \mathrm{d}V = 1$ for any volume that encloses the origin. The temperature at large distances from the origin is zero.

(a) For distances much less than the diffusion length $l = \sqrt{\alpha t}$ the solution may be considered to have reached a steady state. Hence show that the solution for $r \ll l$ is

$$T = \frac{\dot{Q}}{\alpha \rho c_p} \frac{1}{4\pi r} \ .$$

[30%]

(b) Consider the change of variable $\Gamma(r,t) = rT(r,t)$. Show that, if we exclude the origin itself, the governing equation for arbitrary distances from the origin can be rewritten as

$$\frac{\partial \Gamma}{\partial t} = \alpha \frac{\partial^2 \Gamma}{\partial r^2} \ ,$$

and that self-similar solutions of the form $\Gamma = F(r/\sqrt{\alpha t})$ satisfy

$$F''(\eta) + \frac{1}{2}\eta F'(\eta) = 0, \ \text{ where } \ \eta = r/\sqrt{\alpha t} \ .$$

[35%]

(c) Using the results of parts (a) and (b) above, find the temperature distribution, $T(r,t)$, expressed in terms of the integral $\int \exp\left[-\eta^2/4\right] \ \mathrm{d}\eta$. You may use the fact that

$$\int_0^\infty \exp\left[-\eta^2/4\right] = \sqrt{\pi} \ .$$

[35%]

3 We wish to use the Lagrange multiplier method to determine the curve $u(x)$ from $(x,u) = (-1,0)$ to $(1,0)$ that has minimum length and encloses a given (constant) area A with the x-axis (see Fig. 1).

(a) Write down the curve length as an integral $\int_{-1}^{1} F(x,u,u')\,\mathrm{d}x$ in terms of the function $u(x)$. [10%]

(b) Write down the augmented integrand of the constrained functional in the form of $F + \lambda G$, where λ is a parameter and G related to the area constraint. [10%]

(c) Deduce the Euler-Lagrange equation for the augmented functional. [20%]

(d) Deduce the curve $u(x)$ with two integration constants and state what kind of curve it is. [40%]

(e) Use the boundary conditions and the parameter λ to determine the two integration constants in (d). State the valid range for λ. (Note that λ can be determined by the given area A, though we do not ask you to do so here.) [20%]

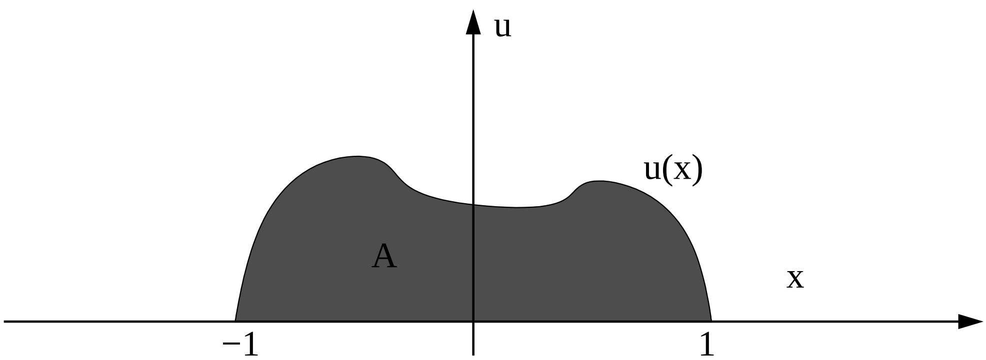

Fig.1

4 (a) Consider a functional of the form

$$I[u(x)] = \int_0^1 F(x, u, u', u'')\ \mathrm{d}x\ ,$$

where $u(x)$ is a smooth function defined in the interval $[0,1]$, and the integrand $F(x, u, u', u'')$ involves the first derivative u' and the second derivative u'' of u. We seek the function $u(x)$ which is a stationary function of $I[u(x)]$.

(i) Write down the directional derivative of the functional $I[u(x)]$. [10%]

(ii) Deduce the governing (Euler-Lagrange) equation for u. [30%]

(iii) Discuss possible boundary conditions of the problem. [10%]

(b) The result in (a) can be used to determine the stationary function $u(x)$ for the functional

$$I[u(x)] = \int_0^1 \frac{1}{2} \left(u''\right)^2\ \mathrm{d}x$$

subject to the boundary conditions

$$u(0) = u'(0) = 0,\ \ u(1) = u'(1) = 0\ ,$$

and the constraint

$$\int_0^1 u\ \mathrm{d}x = 1\ .$$

(i) Write down the augmented integrand of the constrained functional. [10%]

(ii) Deduce the Euler-Lagrange equation for the augmented functional. [10%]

(iii) Use the boundary conditions and the integral constraint to determine the solution of (ii) in (b). [30%]

END OF PAPER

THIS PAGE IS BLANK