Version pad/3

EGT3/EGT2
ENGINEERING TRIPOS PART IIB
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Friday 27th April 2018 2.00 to 3.40

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) The dispersion relationships for deep-water and shallow-water surface gravity waves are $\omega^2 = gk$ and $\omega^2 = ghk^2$, respectively, where ω is the angular frequency, g the acceleration due to gravity, k the wavenumber and h the water depth.

(i) Find the phase and group velocities for both classes of waves. In each case state whether or not the waves are dispersive.

[15%]

(ii) A stone is dropped into a deep pool causing waves to spread radially outward from the point of impact. Sketch the resulting wave pattern, showing the speed of the wave crests relative to that of the overall wave packet.

[15%]

(b) A thin, flat plate of flexural rigidity D and mass per unit area ρ sits on an elastic foundation of stiffness S. Its transverse displacement, $\eta(x, y, t)$, is governed by

$$D\nabla^4\eta + S\eta + \rho\frac{\partial^2\eta}{\partial t^2} = 0,$$

where ∇^4 is the biharmonic operator,

$$\nabla^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2.$$

(i) Derive an expression for the group velocity, $\mathbf{c}_g$, and phase speed, c_p, of flexural waves and hence show that, if $\mathbf{k}$ is the wave-vector,

$$\frac{\mathbf{c}_g}{c_p} = \frac{2Dk^3\mathbf{k}}{Dk^4 + S}, \quad k = |\mathbf{k}| .$$

[30%]

(ii) Show that $|\mathbf{c}_g| > c_p$ when $k > (S/D)^{1/4}$ and $|\mathbf{c}_g| < c_p$ for $k < (S/D)^{1/4}$.

[10%]

(iii) The plate is struck by a hammer and waves spread radially outward from the point of impact. Sketch the resulting wave patterns for $k > (S/D)^{1/4}$ and $k < (S/D)^{1/4}$. Show qualitatively the speed of the wave crests relative to that of the overall wave pattern in each case.

[30%]

2 Consider the one-dimensional heat diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} ,$$

where T is temperature, α the thermal diffusivity, and x the spatial coordinate.

(a) Define the diffusion length and give its physical interpretation. [10%]

(b) Consider the initial condition $T(x, t=0) = T_0 \delta(x)$, where $\delta(x)$ is a unit delta-function centred on the origin and T_0 is a constant. Show that the solution corresponding to this initial condition in the range $-\infty < x < \infty$ is

$$T(x,t) = \frac{T_0}{2\sqrt{\pi \alpha t}} \exp\left[-x^2/4\alpha t\right].$$

You may find it helpful to note that

$$\int_{-\infty}^{\infty} \exp\left(-y^2\right) dy = \sqrt{\pi} .$$

[35%]

(c) Use symmetry to show that, if x_0 is a positive constant,

$$T(x,t) = \frac{T_0}{2\sqrt{\pi \alpha t}} \left[\exp\left(-(x-x_0)^2/4\alpha t\right) + \exp\left(-(x+x_0)^2/4\alpha t\right)\right]$$

is the solution in the domain $0 < x < \infty$ corresponding to the initial condition $T(x, t=0) = T_0 \delta(x - x_0)$ and the boundary condition $\partial T/\partial x = 0$ at $x = 0$. [10%]

(d) Use the result of part (c) plus superposition to find the solution in the domain $0 < x < \infty$ corresponding to the general initial condition $T(x, t=0) = T_0(x)$ and the boundary condition $\partial T/\partial x = 0$ at $x = 0$. Show explicitly that your solution satisfies the initial condition. [30%]

(e) What is the solution in the domain $0 < x < \infty$ corresponding to the general initial condition $T(x, t=0) = T_0(x)$ and the boundary condition $T = 0$ at $x = 0$? [No detailed mathematics is required, but you must justify your answer.] [15%]

3 (a) Let $\mathbf{e}$ be a fixed unit vector and $\mathbf{x}$ the three-dimensional position vector.

(i) Use suffix notation to express

$$(\mathbf{e}\times\mathbf{x})\cdot(\mathbf{e}\times\mathbf{x})$$

in terms of the components of $\mathbf{e}$ and $\mathbf{x}$. [10%]

(ii) Use suffix notation to calculate

$$\nabla\cdot\left[\frac{\mathbf{e}\times\mathbf{x}}{|\mathbf{e}\times\mathbf{x}|}\right]$$

for $\mathbf{e}\times\mathbf{x}\neq\mathbf{0}$. [40%]

(iii) What is the geometrical significance of the vector $(\mathbf{e}\times\mathbf{x})/|\mathbf{e}\times\mathbf{x}|$? [10%]

(b) State Stokes’ theorem and explain carefully all terms related to orientation. [15%]

(c) Let $u(x, y)$ and $v(x, y)$ be two functions in a two-dimensional region S. State the divergence theorem in the two-dimensional case when applied to $\mathbf{B} = (u, v)$. Deduce the divergence theorem from Stokes’ theorem.

[25%]

4 The position of a point on the surface of a sphere of radius *a* can be determined in spherical polar coordinates by the two angles θ and ϕ, where θ is the polar angle measured from the *z*-axis and ϕ the azimuthal angle. A path on the surface of the sphere can be described by the function $\theta = f(\phi)$.

(a) Show that the shortest path between two given points is determined by minimising the integral

$$\int_{\phi_1}^{\phi_2} \sqrt{f'^2 + \sin^2 f}\; d\phi \,,$$

where ϕ_1 and ϕ_2 are the values of ϕ at the end points. [30%]

(b) Use the Euler-Lagrange equation to find a differential equation which must be satisfied by the function $f(\phi)$.

[30%]

(c) Under what condition is $f(\phi) = \theta_0$, where θ_0 is constant, a solution to the equation in part (b)?

[20%]

(d) How can the solutions to part (c) be used to find the shortest path on the surface between two points on the surface?

[20%]

END OF PAPER

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