EGT3/EGT2
ENGINEERING TRIPOS PART IIB
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Friday 26 April 2019 2 to 3.40

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) A wave-bearing system, with dispersion relation $\omega = \omega(k)$, is stimulated to produce a slowly-modulated wave train of the form

$$\eta(x,t) = A\exp\left[i(kx - \omega t)\right],$$

where A, k (and hence ω) are *slowly* varying functions of x and t. This is often rewritten in terms of the phase function, $\theta(x,t)$, as

$$\eta(x,t) = A(x,t)\exp\left[i\theta(x,t)\right].$$

Express the local values of $k(x,t)$ and $\omega(x,t)$ in terms of $\theta(x,t)$ and hence show that

$$\frac{\partial k}{\partial t} + c_g(k)\frac{\partial k}{\partial x} = 0,$$

where $c_g(k)$ is the group velocity. Use this equation to show that an observer must move at the speed $c_g(k)$ to keep seeing waves of wavenumber k. [30%]

(b) How is the group velocity defined in three dimensions and what are its primary physical attributes? [10%]

(c) The governing equation for small-amplitude disturbances in a rapidly-rotating fluid is

$$\frac{\partial^2}{\partial t^2}\nabla^2\mathbf{u} + (2\mathbf{\Omega}\cdot\nabla)^2\mathbf{u} = 0,$$

where $\mathbf{u}$ is the velocity of the fluid in the rotating frame of reference and $\mathbf{\Omega}$ is the background rotation rate. Find the dispersion relationship for such waves and show that the group velocity is

$$\mathbf{c}_g = \pm\frac{2}{|\mathbf{k}|^3}\left[\mathbf{k}\times(\mathbf{\Omega}\times\mathbf{k})\right],$$

where $\mathbf{k}$ is the wavevector. [40%]

(d) A disc is suspended in a rapidly-rotating fluid with its axis aligned with $\mathbf{\Omega}$. At $t = 0$ it starts to oscillate at frequency ω. Sketch the dispersion pattern for $\omega \ll |\mathbf{\Omega}|$ and $\omega = 2|\mathbf{\Omega}|$. [20%]

2 (a) Show that $g(\mathbf{x}) = |\mathbf{x}|^{-1}$ satisfies

$$\nabla^2 g = 0 \quad \text{for } \mathbf{x} \neq 0, \qquad \text{and} \qquad \int_S \nabla g \cdot \mathrm{d}\mathbf{S} = -4\pi,$$

where S is a spherical surface centred on the origin. Hence show that $g(\mathbf{x}) = (4\pi|\mathbf{x}|)^{-1}$ is the solution of

$$\nabla^2 g = -\delta(\mathbf{x})$$

in an infinite domain, where $\delta(\mathbf{x})$ is the three-dimensional delta function. [30%]

(b) Use the results of (a) above to find the general solution to

$$\nabla^2 g = -s(\mathbf{x})$$

in an infinite domain, where s is a source of known distribution. [20%]

(c) The free-space magnetic field, $\mathbf{B}(\mathbf{x})$, induced by a given distribution of steady currents, $\mathbf{J}(\mathbf{x})$, is governed by two of Maxwell's equations:

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Use these equations to show that,

$$\nabla^2 \mathbf{B} = -\mu_0 \nabla \times \mathbf{J},$$

and hence that the form of the magnetic field is given by

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\nabla' \times \mathbf{J}(\mathbf{x}')}{|\mathbf{r}|} \mathrm{d}V', \qquad \mathbf{r} = \mathbf{x} - \mathbf{x}',$$

where ∇' operates on $\mathbf{x}'$ while keeping $\mathbf{x}$ constant. [20%]

(d) Use the vector identity

$$\nabla \times (f\mathbf{A}) = \nabla f \times \mathbf{A} + f \nabla \times \mathbf{A}$$

to show that

$$\nabla' \times \left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{r}|} \right) = \frac{\mathbf{r}}{|\mathbf{r}|^3} \times \mathbf{J}(\mathbf{x}') + \frac{\nabla' \times \mathbf{J}(\mathbf{x}')}{|\mathbf{r}|}$$

and hence use the results of (c) above to establish the Biot-Savert law:

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_0}{4\pi} \int \frac{\mathbf{r} \times \mathbf{J}(\mathbf{x}')}{|\mathbf{r}|^3} \mathrm{d}V'.$$

[30%]

(TURN OVER

3 A uniform inextensible chain, of length l, and total mass m, hangs under gravity between two fixed pegs which are a horizontal distance $L < l$ apart.

(a) If the chain is parameterized by an intrinsic arc-length coordinate $-l/2 < s < l/2$, show that the total gravitational potential energy of the chain is

$$V = \frac{mg}{l} \int_{-l/2}^{l/2} \left(\int_{-l/2}^{s} \sin(\theta(s'))\mathrm{d}s' \right) \mathrm{d}s,$$

where $\theta(s)$ is the angle the chain at s makes with the horizontal. [20%]

(b) Find two integral constraints on the function $\theta(s)$ which ensure the chain hangs between the pegs. [10%]

(c) By minimizing the constrained potential energy, show that the chain adopts the form

$$\theta = \tan^{-1}(As)$$

and explain how you would find the value of A. You do not need to actually find the value of A. [40%]

(d) A second inextensible chain, which also has length l and total mass m, is hung between the pegs. However, this chain has a non-uniform mass per unit length $\rho(s)$. What function, $\rho(s)$, is required for the chain to hang in the arc of a circle with radius R? [30%]

4 (a) Evaluate the following quantities, which are written using (three dimensional) index notation.

(i)

$$\delta_{ij}\delta_{ij}$$

(ii)

$$\varepsilon_{ijk}\varepsilon_{lmn}\varepsilon_{ijk}\varepsilon_{lmn}$$

(iii)

$$\varepsilon_{ijk}\varepsilon_{rjk}$$

[40%]

(b) (i) Compute, from first principles, $DI(y^{(n)})[z]$, the directional derivative of the functional given by:

$$I = \int_0^1 \left(\frac{d^n y}{dx^n}\right)^2 \mathrm{d}x.$$

[10%]

(ii) Find the differential equation and boundary conditions that $y(x)$ must satisfy to minimize I. [20%]

(iii) Find the set of functions $y(x)$ that minimize I subject to the constraint

$$\int_0^1 y(x)\mathrm{d}x = 1,$$

and find the minimum value of I. [30%]

END OF PAPER

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