EGT1
ENGINEERING TRIPOS PART IB

Friday 3 June 2016 2.00 to 4.00

Paper 7

MATHEMATICAL METHODS

Answer not more than ***four*** *questions.*

Answer not more than ***two*** *questions from each section.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

SECTION A

Answer not more than **two** *questions from this section*

1 A vector field is described by $\mathbf{u} = -\frac{Pr}{2}\,\hat{\mathbf{e}}_r + Qz\,\hat{\mathbf{e}}_z$, where P and Q are constants, and $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_z$ are the unit vectors in cylindrical polar coordinates.

(a) Determine the condition for which the flow is solenoidal, and obtain both the scalar potential $\phi(r,z)$ and a vector potential $\mathbf{A} = A(r,z)\,\hat{\mathbf{e}}_\theta$ for that condition. [8]

(b) Obtain an expression for the field lines for general values of P and Q. [5]

(c) Calculate the net flux of $\mathbf{u}$ across the following surfaces which enclose the volume V, indicated in Fig. 1, as a function of geometric parameters, P and Q:

(i) S_1: a circle of radius r_o located on the $z = z_o$ plane. [3]

(ii) S_2: a cylindrical surface of radius r_i and height z_i above the plane $z = 0$, centered on $r = 0$, such that $r_i^2\,z_i = r_o^2\,z_o$. [3]

(iii) S_3: a circle of the same radius r_i at $z = 0$. [2]

(iv) S_4: the conical surface. Express your answer as a function of V , r_o and z_o . [4]

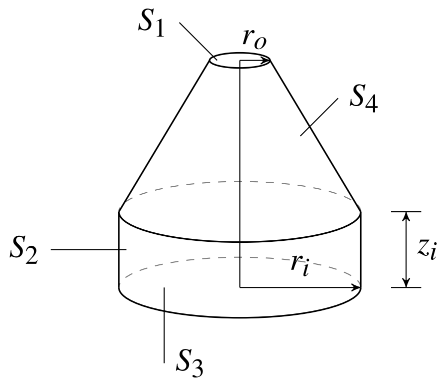

Fig. 1

2 A vector field **B** is expressed in terms of the Cartesian coordinate system (x,y,z) and the associated unit vectors $(\mathbf{i},\mathbf{j},\mathbf{k})$ as

$$\mathbf{B} = yz\frac{df}{dx}\mathbf{i} + fz\,\mathbf{j} + fy\,\mathbf{k}$$

where $f(x)$ is a function only of x.

(a) Show that **B** is a conservative (*i.e.* irrotational) vector field and find its scalar potential ϕ . [7]

(b) The surface S encloses the volume V that is a sphere of radius 1 centred at $(x,y,z) = (0,1,1)$. If the net flux of **B** through S is equal to zero, and if $f(x)$ is differentiable within S, what condition or conditions could $f(x)$ satisfy? [5]

(c) A new vector field is defined as $\mathbf{C} = \psi\mathbf{B}$. If **C** is a conservative vector field, and $\mathbf{B} = \nabla\phi$, find an expression relating $\nabla\psi$ to $\nabla\phi$. Show that this expression is satisfied by $\psi = g(\phi)$, where g is any differentiable function of ϕ. [5]

(d) (i) Evaluate the line integral $\int \mathbf{B}\cdot d\mathbf{l}$ along x from (-1, 1, 1) to (1,1,1) when $f(x) = \sin(\pi x/2)$. [3]

(ii) Using the same function f in item (d)(i), evaluate the line integral $\int \mathbf{B}\cdot d\mathbf{l}$ over the parametric curve given as: [5]

$$x = \cos\theta$$
$$y = \sin\theta + 1$$
$$z = 2 - y$$

for $0 \le \theta \le \pi$.

3 A uniform beam has length L and is aligned along the x-axis, as shown in Figure 2.

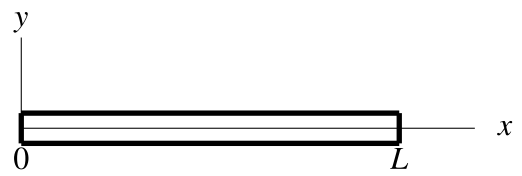

Fig. 2

Small undamped transverse vibrations of this beam are governed by the following partial differential equation for the deflection $y(x,t)$:

$$c^2 \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0$$

where c is a constant.

(a) Assuming that the solution to this equation can be found by separation of variables: $y(x,t) = X(x)T(t)$, find the governing differential equations for X and T. Confirm that the following expressions for X and T satisfy these equations, and find the relationship between ω, k and c. [12]

$$\begin{aligned} X(x) &= A\cos(kx) + B\sin(kx) + C\cosh(kx) + D\sinh(kx) \\ T(t) &= P\cos(\omega t) + Q\sin(\omega t) \end{aligned}$$

(b) The boundary conditions given are:

$$y = 0 \qquad \frac{\partial y}{\partial x} = 0 \qquad \text{at} \quad x = 0$$

$$\frac{\partial^2 y}{\partial x^2} = 0 \qquad \frac{\partial^3 y}{\partial x^3} = 0 \qquad \text{at} \quad x = L$$

Find all solutions to the above governing partial differential equation, subject to these boundary conditions, and assume zero initial velocity, *i.e.* $\frac{\partial y}{\partial t} = 0$ for all x at $t = 0$. Show that the modes are determined by the following relationship: [13]

$$\cosh kL \cos kL = -1$$

SECTION B

Answer not more than **two** *questions from this section*

4 (a) Solve the following system of linear equations using Gaussian elimination:

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + x_4 &= -1 \\ 2x_1 - 2x_2 + x_3 - 2x_4 &= 1 \\ x_1 + x_2 - 3x_3 + x_4 &= 6 \\ 3x_1 - x_2 + 2x_3 - x_4 &= 3 \end{aligned}$$

[2]

(b) Find the **LU** decomposition for the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 3 & 4 \end{pmatrix}$$

[3]

(c) Solve

$$(\mathbf{X} - \mathbf{B})^{-1} = \mathbf{C},$$

in which

$$\mathbf{B} = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$$

[4]

(d) Consider the matrix

$$\mathbf{D} = \begin{pmatrix} 1 & 1 & 3 & a \\ 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \end{pmatrix}$$

Determine a basis for the null space of **D** and state the dimensions of the null space. [8]

(e) If $\mathbf{v}$ is an eigenvector of an invertible matrix $\mathbf{E}$, prove that $\mathbf{v}$ is also an eigenvector of $\mathbf{E}^2$ and $\mathbf{E}^{-2}$, and state the corresponding eigenvalues. [8]

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5 (a) Prove that the following matrix $\mathbf{A}$ is skew-symmetric:

$$\mathbf{A} = \begin{pmatrix} 0 & 6 & -3 \\ -6 & 0 & -8 \\ 3 & 8 & 0 \end{pmatrix}$$

[2]

(b) Find the eigenvectors of the system

$$\mathbf{Bx} = \lambda \mathbf{x},$$

in which

$$\mathbf{B} = \begin{pmatrix} 1 & -1 & 2 \\ -3 & -2 & 3 \\ 2 & -1 & 1 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

[5]

(c) (i) Prove that the matrices $\mathbf{C}\mathbf{C}^T$ and $\mathbf{C}^T\mathbf{C}$ have the same eigenvalues, except for the eigenvalue of 0. [6]

(ii) Let

$$\mathbf{C} = \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

Find all eigenvalues of $\mathbf{C}^T\mathbf{C}$. [4]

(d) Let

$$\mathbf{E} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & -1 & 5 \end{pmatrix}$$

Find an invertible matrix $\mathbf{Z}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{Z}^{-1}\mathbf{E}\mathbf{Z} = \mathbf{D}$$

[8]

6 (a) Suppose x is a continuous random variable with probability density function $f(x) = \frac{1}{2\sqrt{x}}; 1 \leq x \leq 4$. Show that $f(x)$ is a suitable function for a probability density function and calculate the expected value and variance of x. [6]

(b) The breakdown of a conveyor belt is modelled by an exponential distribution with a mean of 25 days. Calculate the probability that the conveyor belt breaks down in a 40 day period. [4]

(c) Assume a failure can always be repaired and repair time can be neglected. Let n be the number of failures of the product over time interval t. From first principles, derive an expression for the probability of a product failing one or more times in time interval t. [10]

(d) The product in part (c) consists of 8 individual parts that all must work for the product to function. The probability of failure of a part is uniformly distributed and is independent of the probability of failure of the other parts. As in part (c), assume a failure can always be repaired and repair time can be neglected. On average each part fails once every 4 years. Calculate the probability of the product failing as a result of a part failing within a time period of 1.5 years. [5]

END OF PAPER

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Answers

1.

(a) $P = Q$

$$\phi = -\frac{Pr^2}{4} + \frac{Pz^2}{2} + \phi_0$$

$$\mathbf{A} = A(r,z) = -\frac{Pzr}{2}\hat{\mathbf{e}}_\theta$$

(b) $r^{2/P} z^Q = \text{const.}$

(c) (i) $Q\pi r_o^2 z_o$

(ii) $-P\pi r_o^2 z_o$

(iii) 0

(iv) $(Q-P)(V - \pi r_o^2 z_o)$

2.

(a) -

(b) $\frac{d^2 f}{dx^2}$ is zero or anti-symmetric about the integration coordinate x

(c) -

(d)(i) 2

(ii) ± 2

3.

(a) $k = \sqrt{\frac{\omega}{c}}$

(b) $\frac{d^2 f}{dx^2}$ is zero or anti-symmetric about the integration coordinate x

(c) $y(x,t) = Y_0 \cos(\omega t)[\cos kx + \cosh kx + R(\sin kx + \sinh kx)]$, where $R = \frac{\sinh kL - \sin kL}{\cos kL + \cos kL}$

4.

(a) $\mathbf{X} = (2, -3, -1, 4)$

(b) $\mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

(c) $\mathbf{X} = \begin{pmatrix} 1.4 & 2.8 \\ 3.3 & 9.1 \end{pmatrix}$

(d) $(-5, 2, 1, 0)$ and $(-(2a-1), -(1-a), 0, 1)$. Null space has dimension 2.

(e) -

5.

(a) -

(b) $-2, -1, 3$

(c) (i)

(ii) $6, 2, 0$

(d) $\begin{pmatrix} -4 & 1 & -8 \\ 1 & 1 & -5 \\ 1 & 1 & 1 \end{pmatrix}$

6.

(a) -

(b) 0.798

(c) $1 - \exp(-n)$

(d) 0.95