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ENGINEERING TRIPOS PART IB

June 2010

Paper 7

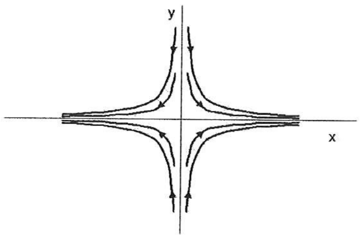
MATHEMATICAL METHODS

Solutions

SECTION A

1 (a) Field lines obey $\frac{dy}{dx} = \frac{u_y}{u_x}$. Therefore:

$$\begin{aligned} \frac{dy}{dx} = \frac{u_y}{u_x} &\Leftrightarrow \frac{dy}{dx} = -\frac{Sy}{Sx} \Leftrightarrow \frac{dy}{y} = -\frac{dx}{x} \\ &\Leftrightarrow \int \frac{dy}{y} = -\int \frac{dx}{x} \Leftrightarrow \ln y = -\ln x + C' \\ &\Leftrightarrow y = \frac{C}{x} \end{aligned}$$



(b) Solenoidal if $\nabla \cdot \mathbf{u} = 0$ and irrotational if $\nabla \times \mathbf{u} = 0$.

$$\nabla \cdot \mathbf{u} = \frac{\partial(Sx)}{\partial x} + \frac{\partial(-Sy)}{\partial y} = S - S = 0$$

$$\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Sx & -Sy & 0 \end{vmatrix} = 0$$

QED

(c) Scalar potential, ϕ, for $\mathbf{u}$ is such that $\mathbf{u} = \nabla\phi$. Therefore:

$$\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} = (Sx)\mathbf{i} - (Sy)\mathbf{j}$$

For the x-component:

$$\frac{\partial \phi}{\partial x} = Sx \Leftrightarrow \phi = \frac{S}{2}x^2 + f_x(x)$$

(cont.

For the y-component:

$$\frac{\partial \phi}{\partial y} = -Sy \Leftrightarrow \phi = -\frac{S}{2}y^2 + f_y(y)$$

so $\phi = \frac{S}{2}(x^2 - y^2) + K$.

The line integral between two points can be found from the difference between the scalar potential at these points and is hence:

$$\int \mathbf{u} \cdot d\mathbf{l} = \phi(\sqrt{2/S}, 1/\sqrt{S}) - \phi(0,0) = \left[\frac{S}{2}\left(\frac{2}{S} - \frac{1}{S}\right) + K\right] - \left[\frac{S}{2}(0-0) + K\right] = \frac{1}{2}$$

(d) The position (X,Y) of the particle obeys $dX/dt = u_x$ and $dY/dt = u_y$, therefore:

$$\frac{dX}{dt} = SX \Leftrightarrow X = X_0 e^{St}$$

$$\frac{dY}{dt} = -SY \Leftrightarrow Y = Y_0 e^{-St}$$

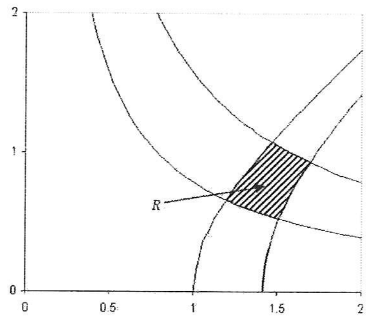
Therefore after a time t the distance r between the two particles released from (X_0, Y_0) and $(X_0 + r_0, Y_0)$ will be:

$r = (X_0 + r_0)e^{St} - X_0 e^{St} \Leftrightarrow r = r_0 e^{St}$. Differentiating w.r.t t gives $dr/dt = Sr_0 e^{St}$, i.e. $dr/dt = rS$. QED.

2 (a) If $r = x$ and $s = y$, then the given identity becomes

$$1 = \frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)} \Leftrightarrow \frac{\partial(x,y)}{\partial(u,v)} = \left[\frac{\partial(u,v)}{\partial(x,y)}\right]^{-1}$$

(b) Curve sketching: consider behaviour as $x, y \to \infty$, find x when $y = 0$ etc.



(c) The transformation $x^2 - y^2 = u$, $2xy = v$ suggests itself. Then, the limiting curves become: $x^2 - y^2 = 1 \Rightarrow u = 1$; $x^2 - y^2 = 2 \Rightarrow u = 2$; $2xy = \pi/2 \Rightarrow v = \pi/2$; and $2xy = \pi \Rightarrow v = \pi$.

To calculate the integral, we need to calculate the Jacobian $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$. Since we do not have explicitly x and y as functions of u and v (we can get such expressions, but the algebra is messy), we use the result from part (a). Therefore:

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ -2y & 2x \end{vmatrix} = 4(x^2+y^2)$$

The integrand becomes: $4(x^4 - y^4)\sin(2xy) = 4(x^2+y^2)\ u\ \sin(v)$. Therefore the integral $I = \int\int 4(x^4 - y^4)\sin(2xy)\, dx\, dy$ becomes:

$$\begin{aligned} I = \int\int 4(x^2+y^2)\ u\ \sin(v)\frac{1}{4(x^2+y^2)}\, du\, dv &= \int_1^2 u du \int_{\pi/2}^{\pi} \sin(v) dv \\ &= \left[\frac{u^2}{2}\right]_1^2 [-\cos(v)]_{\pi/2}^{\pi} \\ &= \frac{3}{2} \end{aligned}$$

NOTE: Some books define the Jacobian matrix $\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$, which has the same determinant as the Jacobian we use in our course.

3 (a) If A is an area enclosing a volume V, and if $\mathbf{n}$ is the normal vector pointing *outwards*, then Gauss's theorem for the heat flux vector $\mathbf{q}$, using $\mathbf{q} = -\lambda \nabla T$, gives for the overall heat flow *out* of V:

$$\iint_A \mathbf{q} \cdot \mathbf{n} \, dA = \iiint_V \nabla \cdot \mathbf{q} \, dV$$
$$= -\iiint_V \lambda \nabla^2 T \, dV$$

But this must be equal to the rate of change (rate of decrease) of the internal energy of the material inside V, or:

$$-\frac{\partial}{\partial t}\left(\rho c_p T V\right) = -\iiint_V \lambda \nabla^2 T \, dV$$

In the limit $V \to 0$, T and $\nabla^2 T$ are uniform inside V, and therefore:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \ , \ \alpha = \frac{\lambda}{\rho c_p}$$

(b) Separation of variables means we seek solutions of the form $T = X(x)\Theta(t)$. By putting in the pde, we get $\alpha X''\Theta = X\Theta'$, or:

$$\frac{\Theta'}{\Theta} = \alpha \frac{X''}{X} = K$$

with K the separation constant and $'$ implying differentiation with respect to the corresponding independent variable. The general form of the solution of these equations is:

$$\Theta = \exp(Kt) \ , \ X = \exp\left(\pm\sqrt{K/\alpha x}\right)$$

Let us explore the nature of these solutions for various values of K. If $K > 0$, then Θ grows exponentially, which is unrealistic for our problem. If $K < 0$, $\Theta \to 0$ at large t for all x, which is unrealistic since the boundary value is oscillating. Therefore let us test $K = i\omega$. In this case,

$$\Theta = \exp(i\omega t) \ , \ X = \exp\left(\pm\sqrt{i\omega/\alpha x}\right)$$

The equation for X needs some attention. Using $\sqrt{i} = (1+i)/\sqrt{2}$, we get: $X = \exp\left[\pm(1+i)\sqrt{\frac{\omega}{2\alpha}x}\right]$, from which we must avoid the e^x behaviour, and therefore $X = \exp\left(-\sqrt{\frac{\omega}{2\alpha}x}\right) \exp\left(-i\sqrt{\frac{\omega}{2\alpha}x}\right)$. Putting everything together:

$$T = T_0 \exp\left[i\left(\omega t - \sqrt{\frac{\omega}{2\alpha}}x\right)\right] \exp\left(-\sqrt{\frac{\omega}{2\alpha}}x\right)$$

(TURN OVER for continuation of Question 3

where use of the boundary condition has been made. Taking the real part, we get the final answer:

$$T(x,t) = T_0 \cos\left(\omega t - \sqrt{\frac{\omega}{2\alpha}}x\right) \exp\left(-\sqrt{\frac{\omega}{2\alpha}}x\right) \qquad (1)$$

Before finishing, let's make some checks. At $x = 0$, $T = T_0 \cos(\omega t)$, as it should. As $x \longrightarrow \infty$, $T \longrightarrow 0$, i.e. the thermal wave does not reach far. (Note that since the heat equation is linear, $T = 0$ is not unrealistic, since we can understand physically T as the temperature above some finite value.) As x increases, the oscillating T lags behind the oscillation at the boundary and the amplitude decreases.

(c) The student is urged to sketch Eq. (1) carefully, for example in Excel or Matlab. The fact that ω appears in both the exponential decay of the amplitude with x and in the cosine describing the oscillation with t implies a special form of the T variation with t and x. An interesting animation of this solution (which is also applicable for other problems, e.g. in the viscous fluid flow above an oscillating plate) can be found at:

`en.wikipedia.org/wiki/Stokes_boundary_layer`

(accessed June 2010).

SECTION B

4 (a) Arrivals are independent, so we have a Poisson distribution with intensity $\lambda = 3000$ per hour.

(b) Arrivals at CPU B are Poisson with $\lambda_B = 1000$ per hour, as the requests are assigned randomly. The approximating Gaussian has mean $\mu = \lambda_B$ and variance $\sigma^2 = \lambda_B$.

(c) Look up cumulative Gaussian at $(1050 - \lambda_B)/\sqrt{\lambda_B} \simeq 1.58$ in data book standard Normal table, which is $p(f|B) = 1 - 0.9429 = 0.0571$.

(d) For CPU A, the similar probability is Gaussian exceeding $(2100 - \lambda_A)/\sqrt{\lambda_A} \simeq 2.24$ which is $p(f|A) = 1 - 0.9875 = 0.0125$ (data book). The probability that the failure was caused by A is given by Bayes' rule:

$$p(A|f) = \frac{p(f|A)p(A)}{p(f|B)p(B) + p(f|A)p(A)} \simeq 0.30.$$

(e) Probability of failure with random assignment $p(f|B)p(B) + p(f|A)p(A) = 0.0274$. Optimal assignment corresponds to a single Poisson with rate 3000 exceeding the total capacity 3150, ie the probability of failure is the prob that std Gaussian exceeds $150/\sqrt{3000} = 2.74$ which is $1 - 0.9969 = 0.0031$ (ie a much lower failure rate).

5 (a) We perform L-U decomposition on the matrix:

$$\mathbf{A}=\begin{bmatrix}1 & 3 & 1 & 4\\ -1 & -1 & 0 & -3\\ 2 & 4 & 1 & 7\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 1 & 4\\ 0 & 2 & 1 & 1\\ 0 & -2 & -1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 2 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 1 & 4\\ 0 & 2 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}=\mathbf{L}\ \mathbf{U}$$

Column space: columns of $\mathbf{L}$ corresponding to non-zeros rows of $\mathbf{U}$, therefore the basis is:

$$\begin{bmatrix}1\\ -1\\ 2\end{bmatrix},\ \begin{bmatrix}0\\ 1\\ -1\end{bmatrix}$$

Row space: The non-zeros rows of $\mathbf{U}$, therefore the basis is:

$$\begin{bmatrix}1\\ 3\\ 1\\ 4\end{bmatrix},\ \begin{bmatrix}0\\ 2\\ 1\\ 1\end{bmatrix}$$

Null space: the solution of $\mathbf{A}\ \mathbf{x}=0$, or equivalently, the solution of $\mathbf{U}\ \mathbf{x}=0$. This gives:

$$\begin{bmatrix}1 & 3 & 1 & 4\\ 0 & 2 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

Using $(x_3,x_4)=(1,0)$ and $(x_3,x_4)=(0,1)$ as two sets of values for the free variables, we get the two basis vectors:

$$\begin{bmatrix}1/2\\ -1/2\\ 1\\ 0\end{bmatrix},\ \begin{bmatrix}-5/2\\ -1/2\\ 0\\ 1\end{bmatrix}$$

Left null space: the solution of $\mathbf{A}^T\ \mathbf{x}=0$. Therefore:

$$\begin{bmatrix}1 & -1 & 2\\ 3 & -1 & 4\\ 1 & 0 & 1\\ 4 & -3 & 7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

(cont.

which gives the vector

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

(b) $\mathbf{A}\ \mathbf{x} = \mathbf{b} \Leftrightarrow \mathbf{LU}\ \mathbf{x} = \mathbf{b} \Leftrightarrow \mathbf{L}\ \mathbf{c} = \mathbf{b}$, where $\mathbf{U}\ \mathbf{x} = \mathbf{c}$. For a solution to exist, $\mathbf{b}$ must be in column space, therefore $\mathbf{b}$ must be normal to every vector in null space. Therefore

$$\mathbf{b} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = 0 \Rightarrow -1 + a = 0 \Rightarrow a = 1$$

(c) The general solution will be the particular solution plus a linear combination of the null space basis vectors. To find the particular solution, we first find $\mathbf{c}$ given by $\mathbf{Lc} = \mathbf{b}$, which gives $\mathbf{c} = [1\ 1\ 0]^T$, and then we solve for $\mathbf{Ux} = \mathbf{c}$. Therefore:

$$\begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(putting zero's for the free variables), which gives $x_1 = -1/2$ and $x_2 = 1/2$. Therefore the general solution is:

$$\mathbf{x} = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}$$

By performing the multiplication $\mathbf{Ax}$, it is easy to show that $\mathbf{Ax} = \mathbf{b}$.

6 (a) Provided the eigenvectors of $\mathbf{A}$ span $\Re^N$, any vector $\mathbf{x}$ can be written as $\mathbf{x} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots a_N\mathbf{u}_N$, where $\mathbf{u}_1$ etc are the eigenvectors with corresponding eigenvalues $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_N|$. Then, $\mathbf{A}^k\mathbf{x} = a_1\lambda_1^k\mathbf{u}_1 + a_2\lambda_2^k\mathbf{u}_2 + \ldots + a_N\lambda_N^k\mathbf{u}_N$. For large k, the first terms dominate, so $\mathbf{A}^k \approx a_1\lambda_1^k\mathbf{u}_1 + a_2\lambda_2^k\mathbf{u}_2$, which implies

$$\frac{|\mathbf{A}^{k+1}\mathbf{x}|}{|\mathbf{A}^k\mathbf{x}|} \approx \lambda_1$$

The rate of convergence is determined by $|\lambda_1/\lambda_2|$.

(b) The matrix $(\mathbf{A} - 3\mathbf{I})^{-1}$ has eigenvalues $(\lambda_i - 3)^{-1}$, where λ_i are the eigenvalues of $\mathbf{A}$. The closest eigenvalue to 3 will be the largest (in magnitude) eigenvalue of $(\mathbf{A} - 3\mathbf{I})^{-1}$. We expect the convergence rate to be:

$$\left|\frac{\lambda' - 3}{\lambda - 3}\right|$$

with λ the nearest eigenvalue to 3 and λ' the next nearest.

(c)

$$(\mathbf{B} - 1\mathbf{I})^{-1} = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 10 \\ 10 & 0 \end{bmatrix}$$

Therefore:

$$(\mathbf{B} - \mathbf{I})^{-1}\mathbf{x} = \begin{bmatrix} 0 & 10 \\ 10 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10x \\ 10y \end{bmatrix}$$

and

$$(\mathbf{B} - \mathbf{I})^{-2}\mathbf{x} = \begin{bmatrix} 100x \\ 100y \end{bmatrix}$$

which evidently means that $(\mathbf{B} - \mathbf{I})^{-1}$ has eigenvalue 10, $(\mathbf{B} - \mathbf{I})^{-2}$ has eigenvalue 100, etc., therefore the eigenvalues of $\mathbf{B}$ are 1 ± 0.1, i.e. 0.9 and 1.1. (Note: by performing the calculation of the eigenvalues of $\mathbf{B}$ one finds the same values.) The procedure does not work for the eigenvectors because it enters a cycle.

(d) The inverse power method implies $\mathbf{y}_k = (\mathbf{A} - \alpha\mathbf{I})^{-1}\mathbf{y}_{k+1}$, with $\mathbf{y}_k = (\mathbf{A} - \alpha\mathbf{I})^{-k}\mathbf{x}$. It is computationally cheaper to solve $(\mathbf{A} - \alpha\mathbf{I})\mathbf{y}_{k+1} = \mathbf{y}_k$ with an LU factorisation.

END OF PAPER