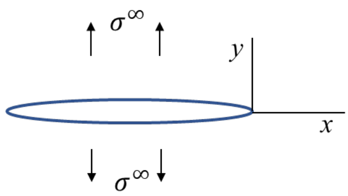
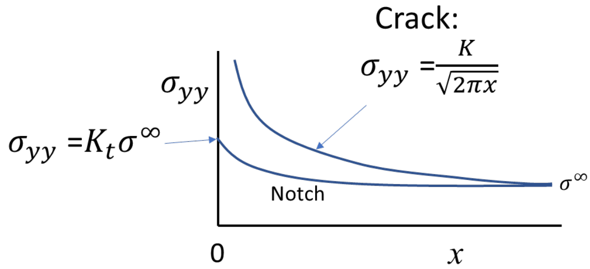
Part IIA Engineering Tripos, 2022

Crib for 3C9 : Fracture Mechanics of Materials and Structures.

Q1. (a) The magnitude of the stress ahead of a crack tip scales with the stress intensity factor, K where K is given by $K = \sigma^{\infty}\sqrt{\pi a}$ for an isolated crack of length $2a$ in an infinite sheet. The enhanced stress ahead of the crack (or notch) exists over a length of about a/10. In Linear Elastic Fracture Mechanic (LEFM), it is assumed that fracture occurs when K attains the fracture toughness K_{IC} of the solid, a material property. Irwin showed that K is directly related to the energy release rate, $\mathcal{G}$. by $E\,\mathcal{G} = K^2$. The energy release rate is the energy released to the crack tip per unit area of new crack. If this equals the material toughness $\mathcal{G}_{IC}$ then the crack advances. The two material properties s K_{IC} and $\mathcal{G}_{IC}$ are also related by the Irwin relation, $E\,\mathcal{G}_{IC} = K_{IC}^2$. [15%]

(b) The elliptical notch has a finite stress concentration K_t at its tip, whereas the crack has an infinite stress concentration factor.





The stress ahead of a notch may be sufficient to yield the solid: if $K_t\sigma^{\infty}$ exceeds the yield strength. [15%]

(c) (i) Shear modulus of epoxy is $G=0.5E/(1+\nu)$.

Write σ as the tensile stress in the adhesive layer due to a tensile strain $\varepsilon = w/h$ and note that the tangential strain in the adhesive (parallel to the interface) =0. Also, the stress normal to the face of the adhesive layer vanishes. Consequently, $\sigma = E\varepsilon/(1-\nu^2)$.

Write τ as the shear stress in the adhesive layer due to a tensile strain $\gamma = u/h$, with $\tau = G\,\gamma$

The strain energy density $=0.5\sigma\varepsilon + 0.5\tau\,\gamma$

implying that the total strain energy T for the adhesive layer of unit thickness is

$$T = Gh(W-a)\left(\frac{u}{h}\right)^2 + \frac{E}{(1-\nu^2)}h(W-a)\left(\frac{w}{h}\right)^2$$

This assumes that there is no stress elevation at the crack tip. Note that this stress state does not give zero traction at the free ends of the adhesive layer. [30%]

(ii) The potential energy of the dead loads *S* and *N* are –(*Su*+*Nv*)

And so the potential energy *P* of the SYSTEM (adhesive layer and dead loads) is

$$P = T - (Su + Nw) = -W - \frac{hS^2}{4G(W-a)} - \frac{(1-\nu^2)}{4E}\frac{hN^2}{(W-a)}$$

Energy release rate $\mathcal{G}$ is

$$\mathcal{G} = -\frac{\partial P}{2\partial a} = \frac{hS^2}{8G(W-a)^2} - \frac{(1-\nu^2)}{8E}\frac{hN^2}{(W-a)^2}$$

The presence of a positive mode II component (due to the finite shear force S) causes the right-hand crack tip to kink down to the lower interface, and the left-hand crack tip to kink up to the upper interface. The cracks then grow as interfacial cracks.

[40%]

Q2. (a) Consider first the case of cleavage of an elastic-brittle solid. Neglect the role of crack tip plasticity. When a mode II crack in an elastic solid undergoes crack growth it does so by the formation of an inclined kink along a pure mode I path, such that the tip of the kink suffers pure mode I loading. There is a small drop in K_I at the kink tip compared to that of the initial planar crack. Crack advance demands that the the value of K_I at the kink tip equals K_{IC}. Consequently, the observed mode II fracture toughness is comparable (actually slightly greater than) the mode I value.

Second, consider the case of ductile fracture. Under mode I loading, a high triaxial state exists ahead of the crack tip, promoting void growth and a low value of fracture toughness. In contrast, under mode II loading, no such elevation in hydrostatic stress occurs and so void growth and coalescence requires a larger value of stress intensity factor. Consequently, the mode II fracture toughness can significantly exceed the mode I value. [40%]

(c) (i) A single arm displaces by *u* under an end load *P*. Beam theory tells us:

$$u = \frac{P\ell^3}{3EI}$$

where

$$I = \frac{1}{12}Bh^3$$

Compliance:

$$C = \frac{2u}{P} = \frac{2a^3}{3EI} = \frac{8a^3}{EBh^3}$$

Energy release rate:

$$\mathcal{G} = \frac{1}{2}P^2\frac{\partial C}{B\partial a} = P^2\frac{12a^2}{EB^2h^3}$$

	$K = \sqrt{E\mathcal{G}} = \frac{2\sqrt{3}Pa}{Bh^{3/2}}$	[30%]

(ii) The yield moment of the cross-section is $M_y = \frac{1}{4}\sigma_y Bh^2$

and so the limit load is $P_L = M_y/a$

The stored energy for the 2 arms is $W = 2P_L u_0 = \frac{1}{2}\sigma_y Bh^2 u_0/a$

and so

$$J = -\frac{\partial W}{\partial a} = \frac{1}{2}\sigma_y Bh^2 u_0/a^2$$

[30%]

Q3. (a) First calculate the residual value of *K*, termed K_R, due to the residual tensile stress $\sigma_R = \sigma_Y$ such that

$$K_R = \sigma_Y\sqrt{\pi a} \qquad \text{for a < w}$$

and

$$K_R = \frac{2\sigma_Y w}{\sqrt{\pi a}}\frac{a}{w}\sin^{-1}\frac{w}{a}$$

for a > w.

The working stress $\sigma_W = -\sigma_Y/2 \rightarrow 0$ gives rise to

$$K_W = -\frac{\sigma_Y}{2}\sqrt{\pi a} \rightarrow 0$$

for all *a*.

Hence, for a<w,

$$K_{min} = \frac{\sigma_Y}{2}\sqrt{\pi a}$$

$$K_{max} = \sigma_Y\sqrt{\pi a}$$

$$\Delta K = \frac{\sigma_Y}{2}\sqrt{\pi a}$$

and for a > w,

$$K_{min} = -\frac{\sigma_Y}{2}\sqrt{\pi a} + \frac{2\sigma_Y w}{\sqrt{\pi a}}\frac{a}{w}\sin^{-1}\frac{w}{a}$$

$$K_{max} = \frac{2\sigma_Y w}{\sqrt{\pi a}} \frac{a}{w} \sin^{-1} \frac{w}{a}$$

$$\Delta K = \frac{\sigma_Y}{2} \sqrt{\pi a}$$

[30%]

(b) Calculate the value of a/w for which

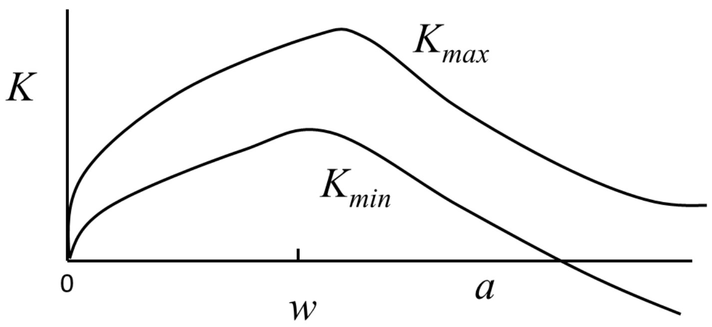
$$K_{min} = 0 = -\frac{\sigma_Y}{2} \sqrt{\pi a} + \frac{2\sigma_Y w}{\sqrt{\pi a}} \frac{a}{w} \sin^{-1} \frac{w}{a}$$

Direct evaluation gives $a/w = 1/\sqrt{2}$

At $a \gg w$, we have $K_{\min} < 0$ and

$$K_{max} \to \frac{2\sigma_Y w}{\sqrt{\pi a}}$$

since $\frac{a}{w} \sin^{-1} \frac{w}{a} \to 1$



[30%]

(c) Make use of

$$\Delta K = \frac{\sigma_Y}{2} \sqrt{\pi a}$$

for all a. Then,

$$\frac{dN}{da} = \frac{1}{C} \left(\frac{2}{\sigma_y \sqrt{\pi}} \right)^n a^{-n/2}$$

which integrates for $a = w/10$ to $a = w$, to give

$$N = \frac{1}{C}\left(\frac{2}{\sigma_y\sqrt{\pi}}\right)^n \frac{2}{2-n}\left[w^{(2-n)/2} - (w/10)^{(2-n)/2}\right]$$

[30%]

(d) The crack will stay closed due to stress relief, and so the fatigue life is infinite. [10%]

Q4. (a) The aluminium alloy has a lower yield strength than glass due to the lower resistance to dislocation motion. The transition flaw size of the Al alloy is on the order of serval mm, and so the aluminium alloy flows plastically. Its ductility is dictated by its strain hardening exponent, as revealed by the Considere construction.

Silica glass has a much higher yield strength and a much lower toughness than the aluminium alloy, and has a transition flaw size on the order of microns. Consequently, the bar behaves in an elastic, brittle manner with the strength dictated by its fracture toughness and the largest flaw size. The plastic strain is negligible at fracture. [25%]

(b) The main toughening mechanism is the pull-put of fibres in the wake of the growing crack. No such pull-out mechanism occurs for a crack in the epoxy or in carbon alone. The pull-out mechanism relies upon a statistical distribution of flaws along the length of each carbon fibre, leading to fibre fracture off the main cracking plane. The fractured fibres pull out from their sockets and this is a potent toughening mechanism by crack bridging. The longer the pull-out length the tougher is the composite. [25%]

(c) Recall that the toughness of a thin sheet in plane stress exceeds that of a thick sheet in plane strain. Plane strain conditions give rise to a higher hydrostatic stress state and thereby promotes void growth and coalescence at the crack tip. To achieve plane stress within each sheet, local debonding must occur between thin sheets in the vicinity of the crack tip, and the interfacial fracture elevates the toughness. [25%]

(d) J-integral tests can be performed on small scale bending specimens, for which no outer K-field exists. The measured J value can be translated directly into a fracture toughness value via the Irwin relation, $E\,J_{IC} = K_{IC}^2$. The size criterion for a J-integral bend test is much less severe than that for a K test. A J versus crack extension Δa R-curve test can be performed on a small 3 point bend specimen and converted immediately into a K versus Δa R-curve response for the material by making use of $E\,J = K^2$. [25%]